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1987 J. Phys. A: Math. Gen. 20 L221

(http://iopscience.iop.org/0305-4470/20/4/008)

J. Phys. A: Math. Gen. **20** (1987) L221-L226. Printed in the UK

LETTER TO THE EDITOR

D_n models: local densities

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Received 17 November 1986

Abstract. We compute the local densities of the D_n series, whose integrability was shown in a previous letter.

In an earlier letter (Pasquier 1987) we showed the complete integrability of a set of models labelled by D_n. Here we continue the analyses and compute the height probabilities in regime III. They can be straightforwardly derived from expressions in an RSOS model with $r=2n-2$ by applying a reflection principle (Andrews *et al* 1984). We thus confirm the conjecture that these models describe universality classes of unitary conformal theories (Pasquier 1986).

We recall results from our previous letter (Pasquier 1987): D_n models are IRF models defined in terms of even and odd heights lying on sites of two interpenetrating sublattices. We require that two heights can be adjacent only if they are linked on the D_n diagram (figure 1).

Using the same notation as Andrews *et al* (1984), we define

$$\begin{aligned} \alpha_l &= \theta_1(\mu-u)/\theta_1(\mu) \\ \beta_l &= \theta_1(u)(\theta_1(\mu))^{-1}\left(\frac{\theta_2((l-1)\mu)\theta_2((l+1)\mu)}{\theta_2^2(l\mu)}\right)^{1/2} \\ \gamma_l &= \theta_2(l\mu+u)/\theta_2(l\mu) \\ \delta_l &= \theta_2(l\mu-u)/\theta_2(l\mu) \\ \mu &= \pi/2(n-1). \end{aligned} \tag{1}$$

There are two kinds of weights, those for which none of the heights around the face takes the value $0, \bar{0}$:

$$\begin{aligned} W(l, l+1|l-1, l) &= W(l, l-1|l+1, l) = \alpha_l \\ W(l+1, l|l, l-1, l) &= W(l-1, l|l, l+1) = \beta_l \\ W(l+1, l|l, l+1) &= \gamma_l \\ W(l-1, l|l, l-1) &= \delta_l \end{aligned} \tag{2}$$

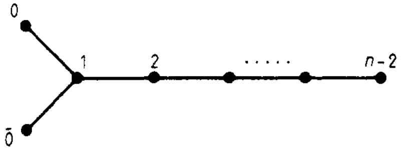

Figure 1. D_n.

with $l \geq 2$ in α_l, β_l, δ_l; $l \geq 1$ in δ_l, and weights for which some heights can take the value $0, \bar{0}$:

$$\begin{aligned}
&W(1,2|0,1)=W(1,2|\bar{0},1)=W(1,0|2,1)=W(1,\bar{0}|2,1)=\alpha\\
&W(1,\bar{0}|0,1)=W(1,0|\bar{0},1)=\bar{\alpha}\\
&W(2,1|1,0)=W(0,1|1,2)=W(2,1|1,\bar{0})=W(\bar{0},1|1,2)=\beta\\
&W(0,1|1,\bar{0})=W(\bar{0},1|1,0)=\bar{\beta}\\
&W(1,0|0,1)=W(1,\bar{0}|\bar{0},1)=\gamma\\
&W(0,1|1,0)=W(\bar{0},1|1,\bar{0})=\delta
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\alpha=\alpha_1\\
&\beta=\frac{1}{\sqrt{2}}\beta_1\\
&\bar{\beta}=\tfrac{1}{2}(\delta_1-\alpha_1)\\
&\delta=\tfrac{1}{2}(\delta_1+\alpha_1)\\
&\gamma=(\gamma_0+\Gamma\beta_1)\\
&\bar{\alpha}=(\gamma_0-\Gamma\beta_1)\\
&\Gamma=\theta_2^2(\mu)\left(\frac{1}{\theta_2(2\mu)\theta_2^3(\mu)}\right)^{1/2}.
\end{aligned} \tag{3'}$$

These weights obey the diagonal reflection symmetry conditions:

$$W(a,b|c,d)=W(d,b|c,a)=W(a,c|b,d). \tag{4}$$

They depend on two parameters, u and p, the nome of elliptic functions entering their definitions. Here we will restrict ourselves to regime III:

$$\begin{aligned}
&0\leq u\leq\mu\\
&|p|\leq 1.
\end{aligned} \tag{5}$$

We regard p as a given constant and u as a variable. The function W has the rotation and inversion properties:

$$\begin{aligned}
&W_u(l_1l_2|l_3l_4)=\left(\frac{g_{l_1}g_{l_4}}{g_{l_2}g_{l_3}}\right)^{1/2}W_{\mu-u}(l_3l_1|l_4'l_2)\\
&g_l=\theta_2^{1/2}(l\mu)\qquad l\geq 1\\
&g_0=g_{\bar{0}}=\frac{1}{\sqrt{2}}\theta_2^{1/2}(0)
\end{aligned} \tag{6}$$

$$\sum_l W_u(l_1l_2|l_3l)\,W_{-u}(ll_2|l_3l_4)=\delta_{l_1l_4}\frac{\theta_1(\mu-u)\theta_1(\mu+u)}{\theta_1^2(\mu)}. \tag{7}$$

We now proceed to compute the local probabilities

$$P_a=Z^{-1}\sum\delta(l_1,a)\prod_{(ijnm)}W(l_i,l_j|l_m,l_n) \tag{8}$$

where Z is the partition function. The sum is over all allowed arrangements of heights on the lattice. We will mainly follow the procedure of appendix A of Andrews *et al* (1984) with some modifications adapted to our case. Define a state $|\boldsymbol{l}\rangle = |l_1 \ldots l_m\rangle$ by a set of heights on the lattice. Starting from the centre site l_1 and moving right to the boundary, $l_{m+1} = b$, $l_{m+2} = c$ are fixed by the ground state chosen for the boundary condition.

Following Andrews *et al* one has

$$P_a = \mathrm{Tr}\, S_a R_1^2 \exp(-2\mu H)/\mathrm{Tr}\, R_1^2 \exp(-2\mu H). \tag{9}$$

S_a projects on height a at site 1:

$$\begin{aligned} S_a|\boldsymbol{l}\rangle &= \delta_{l_1,a}|\boldsymbol{l}\rangle \\ R_1|\boldsymbol{l}\rangle &= g_{l_1}|\boldsymbol{l}\rangle. \end{aligned} \tag{10}$$

$\exp(-\mu H)$ is the diagonal form of the corner transfer matrix A defined in Baxter (1982).

To diagonalise A, it is convenient to make a conjugate modulus transformation. Define

$$f(W, q) = \prod_{n=1}^{\infty} (1-q^{n-1}W)(1-q^n W)(1-q^n). \tag{11}$$

It is straightforward to express the weights in terms of f:

$$\begin{aligned}
\alpha_l &= \left(\frac{\lambda_l\lambda_l}{\lambda_{l-1}\lambda_{l+1}}\right) W^{1/2}\frac{f(xW^{-1})}{f(x)} \\
\gamma_l &= \left(\frac{\lambda_{l+1}\lambda_{l+1}}{\lambda_l\lambda_l}\right)\frac{f(x^{(n-1+l)}W)}{f(x^{(n-1+l)})} \\
\delta_l &= \left(\frac{\lambda_{l-1}\lambda_{l-1}}{\lambda_l\lambda_l}\right)\frac{f(x^{(n-1+l)}W^{-1})}{f(x^{(n-1+l)})} \\
\beta_l &= \left(\frac{\lambda_l\lambda_l}{\lambda_{l-1}\lambda_{l+1}}\right)\left(\frac{xf(x^{(n+l-2)})f(x^{(n+l)})}{f^2(x^{(n+l-1)})}\right)^{1/2}\frac{f(W)}{f(x)} \\
\delta, \bar{\beta} &= \left(\frac{\lambda_1\lambda_1}{\lambda_0\lambda_0}\right)\frac{1}{2}\left(\frac{f(x^nW^{-1})}{f(x^n)} \pm W^{1/2}\frac{f(xW^{-1})}{f(x)}\right) \\
\gamma, \bar{\alpha} &= \left(\frac{\lambda_0\lambda_0}{\lambda_1\lambda_1}\right)\left(\frac{f(x^{n-1}W^{-1})}{f(x^{n-1})} \pm x^{1/2}W^{-1/2}\frac{f(x^n)}{f(x^{n-1})}\right)
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\theta_1(u,p) &= \theta_1(u, \mathrm{e}^{-\varepsilon}) \\
&= \frac{1}{2}\left(\frac{2\pi}{\varepsilon}\right)^{1/2} \exp\left(\frac{\varepsilon}{8} - \frac{\pi^2}{8\varepsilon} - \frac{2u(\pi-u)}{\varepsilon}\right) f\left(\exp\left(-\frac{4\pi u}{\varepsilon}\right), \exp\left(-\frac{4\pi^2}{\varepsilon}\right)\right) \\
W &= \exp(-4\pi u/\varepsilon) \\
x &= W(\mu) \\
f(W) &= f(W, x^{2(n-1)}) \\
\lambda_l &= \exp(-\tfrac{1}{2}l^2\mu u/\varepsilon).
\end{aligned} \tag{13}$$

The diagonal form of $\exp(-uH)$ then takes the following form (formula (A40) of Andrews *et al*):

$$\exp(-uH) = \lambda_{l_1}^{-1} W^{N(l)/2} \delta(\boldsymbol{l}, \boldsymbol{l}'). \tag{14}$$

$\boldsymbol{l}$, $\boldsymbol{l}'$ denotes two sets of heights $\{l_1 \ldots l_m\}\{l'_1 \ldots l'_m\} l_{m+1}$, l'_{m+1}, l_{m+2} are fixed by their boundary values $l_{m+1} = l'_{m+1} = b$, $l_{m+2} = c$.

$N(l)$ is an integer independent of x which can be obtained by looking at the limit $x = 0$. The main difference with the Andrews *et al* analyses is that matrices U_j defined in A2 of their paper are not always diagonal in this limit.

If one defines the matrices:

$$U_{l_j l'_j}^{(l_{j-1}, l_{j+1})} = W(l_j, l_{j+1} | l_{j-1}, l'_j). \tag{15}$$

They become diagonal except if $l_{j-1} = l_{j+1} = 1$, l'_j, $l_j = 0$ or $\bar{0}$, then

$$U^{(1,1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & W^{1/2} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{16}$$

Replace $0, \bar{0}$ by two new 'heights'

$$X, Y = \frac{1}{\sqrt{2}}(0 \pm \bar{0}). \tag{17}$$

Then $N(\bar{l})$ takes the following form:

$$N(\bar{l}) = \sum_{j=1}^{m} j d_j(l_{j+2}, l_j)/2 \tag{18}$$

with

$$\begin{aligned} &d_j(l_{j+2}, l_j) = |l_{j+2} - l_j| && \text{if} \quad l_{j+2} \text{ or } l_j \neq 1 \\ &d_j(1,1) = 0 && \text{if} \quad l_{j+1} = X \\ &d_j(1,1) = 2 && \text{if} \quad l_{j+1} = Y \\ &d_j(X, Y) = 0 \\ &d_j(X, 2) = d_j(Y, 2) = 2. \end{aligned} \tag{19}$$

It is now easy to map the problem into an RSOS one with $r = 2n - 2$ (A_{2n-3} with our notation). To a set of 'heights' $l_1 l_2 \ldots l_m$ with some of the heights equal to X or Y, associate a new set of heights $l'_1 l'_2 \ldots l'_m$ with $l'_i = 0$ if $l_i = X$ or Y and $l'_i = (-1)^{\Phi(i)} l_i$ if $l_i = 1, 2, \ldots, n-2$. $\Phi(i)$ is the number of heights $l_j = Y$ with $j < i$. The heights l'_i are integers between $-n-2$ and $n+2$ subject to the constraint $|l'_{j+1} - l_j| = 1$. One then has

$$N(\boldsymbol{l}) = \sum_{j=1}^{m} j |l'_{j+2} - l'_j|/2 \tag{20}$$

which is the expression from Andrews *et al* for $N(\boldsymbol{l})$ (formula 1.5.2 of their paper). The calculation is then the same as theirs, the only difference being that a ground state, $l_{m+1} = b$, $l_{m+2} = c$, has to be identified with its reflected

$$l_{m+1} = -b \qquad l_{m+2} = -c.$$

The results can now be obtained from theirs. In formula (3.3.18c), they express P_a in terms of d which determines the system in the ordered regime. Adding the two contributions coming from the two reflected ground states we thus obtain

$$P_a = \left[\theta_3\left(\frac{\pi a}{4(n-1)}+\frac{\pi d}{2(2n-3)}, S\right) + \theta_3\left(\frac{\pi a}{4(n-1)}-\frac{\pi d}{2(2n-3)}, S\right) - \theta_4\left(\frac{\pi a}{4(n-1)}+\frac{\pi d}{2(2n-3)}, S\right) - \theta_4\left(\frac{\pi a}{4(n-1)}-\frac{\pi d}{2(2n-3)}, S\right)\right] \times \left[R_a \theta_4(0, p^{2(n-1)})\theta_2\left(\frac{\pi d}{2n-3}, S^{8(n-1)}\right)\right]^{-1}$$

for $a \geqslant 1$. $P_0 + P_{\bar{0}} = \frac{1}{2}$ (the above expression with $a=0$)

$$S = p^{1/4(2n-3)} \tag{21}$$

$$R_a = 2(n-1)[\theta_2(\pi a/2(n-1), P)]^{-1}. \tag{22}$$

P is the parameter entering the definition of the weights; d determines the phase of the ordered regime

$$d = \tfrac{1}{2}(b+c). \tag{23}$$

Note that the phase with $b=0$, $c=1$ (or $b=1$, $c=0$) is not a true phase, but rather a coexistence phase of the two phases $b=0$, $c=1$ and $b=\bar{0}$, $c=1$.

From (6) and (7), the partition function per site $K(u)$ obeys the same inversion relations as the partition function of the RSOS model with $r = 2n-2(A_{2n-3})$. It is therefore equal to it.

We consider the limit $p \to 0$, where p measures the deviation from criticality. The free energy vanishes as

$$K_{\text{sing}} = p^{2-\alpha} \log p \tag{24}$$

with

$$2-\alpha = n-1. \tag{25}$$

We expand the expressions of P_a around the critical point:

$$P_a = \frac{2\cos(\pi a/2(n-1))}{(n-1)\cos(\pi d/(2n-3))} \sum_{\substack{m=1 \\ m \text{ odd}}}^{2n-3} p^{(m^2-1)/8(2n-3)} \cos\left(\frac{m\pi a}{2(n-1)}\right)\cos\left(\frac{m\pi d}{2n-3}\right)(1+\mathrm{O}(p)) \tag{26}$$

for $a \geqslant 1$ and $P_0 + P_{\bar{0}}$ is equal to one-half of the preceding expressions with $a=0$.

Repeating the analysis of Huse (1984), it is possible to build order parameters with a definite scaling dimension

$$x_m = \frac{m^2-1}{2(2n-2)(2n-3)} \tag{27}$$

for

$$m = 1, 3, \ldots, 2n-5, n-1$$

$$[m] = P_0 + P_{\bar{0}} + \sum_{a=1}^{n-2} \cos\left(\frac{m\pi a}{2(n-1)}\right)\frac{P_a}{\cos(\pi a/2(n-1))} \tag{28}$$

$$[n-1] = P_0 - P_{\bar{0}}.$$

These operators were identified with the highest weights of Virasoro representations $[m, m]\otimes[m, m]$ in Pasquier (1986). The values of m correspond to those of the models (A_{2n-4}, D_n) of Cappelli *et al* (1986).

Unfortunately, the above analysis does not give access to the scaling dimension of the odd operator under the exchange symmetry of 0 and $\bar{0}$: $P_0 - P_{\bar{0}}$. For this, the result above is still a conjecture.

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